Elastic scattering and energy theorems for a doubly periodic planar array of elastic obstacles

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Mixed vector-dyadic equations derived earlier [V. Twersky, J. Math. Phys. **16**, 633 (1975)] for multiple scattering of elastic waves are applied to a doubly periodic planar array of elastic obstacles. Different approximations for the multiple-scattering wave and amplitudes are given. Three groups of four energy theorems are discussed. Results for identical rigid elastic scatterers are derived.

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I. INTRODUCTION

We consider the multiple scattering of elastic waves by a doubly periodic planar array of bounded elastic obstacles and follow the procedure of the work of Twersky in Ref. [1] for the doubly periodic planar array of bounded acoustical objects.

In Ref. [1], Twersky based his treatment on plane-wave integral forms of the scattered field and of the multiple-scattering amplitude. He showed that they lead directly to the array-mode functional representation in terms of the single-scattering amplitude, and to single approximations for the array resonance.

Here, we extend the results of Twersky into elasticity by exploiting the derivations and the mixed vector-dyadic formalism of Refs. [2–4]. We preserve, whenever possible, the notation, the terminology, and the equational forms of Ref. [1]. We modify and transform the appropriate operators to reflect and support the tensorial nature of the present elastic problem.

From general reciprocity relations of Refs. [5] and [6], we establish three groups of four energy theorems adapted to the geometry of the array for multiple-scattering amplitudes separately, multiple- and single-scattering amplitudes together, and single-scattering amplitudes alone. The first group relates the outside multiple fields of the array, the second represents energy-conservation requirements involving the multiple field and the net outside field of the single scatterer, and the third regulates the single-scattered fields in the array for two arbitrary directions of incidence at the surface of the central scatterer.

The second group of energy theorems, the mixed multiple-field–single-field interactions, is important to inverse elastic scattering as in Ref. [7] where a generalized optical theorem for elastodynamics derived from the Newton-Marchenko equation is presented. The second group of energy theorems provides a way of predicting single-scattering results from the knowledge of the multiple-scattering field. Therefore, it is useful to modeling and simulation activities especially when dealing with inhomogeneous layers of very near-identical or nonidentical scatterers (Ref. [8]).

In addition to the leading term approximation of the multiple-scattering amplitudes and of the energy theorems, we study the case of Rayleigh scattering for identical rigid elastic spheres. The results are given in forms ready for application which contain only the known single-scattering amplitudes of the single object in isolation. Needed formalism and required structure for the energy theorems in the Rayleigh regime are established. Specific Rayleigh results are obtained for multiple elastic theorems in the forward direction of scattering corresponding to two arbitrary longitudinal directions of incidence.

The results presented in this paper are crucial to the study of the low-frequency coupling of the elastic double lattice. They will play a significant role in atmospheric optics (Ref. [9]) and in the application of scattering theory to planetary atmospheric studies (Ref. [10]). They will certainly contribute to the still-open question of stability in elastodynamic (Ref. [11]) and to ultrasonic detection and characterization of flaws in metals and ceramics (Ref. [12]). Other applications of this problem (with its relativistic complement) can be found in galactic scattering by comet dust particle (Ref. [13]).

Due to the difficult nature of the governing elastic equations for this problem, the coupled stress transition conditions at the surface of the scatterers, and the complicated nature of the resulting elastic lattice from the doubly periodic planar array, the special functions representation of the multiple mixed-vector dyadic scattering amplitudes and the analysis of boundary dependent low-frequency coupling effects are left for a separate study.

Because of mode conversion at the boundary of each scatterer, transverse or s waves and longitudinal or p waves always appear together. We use $\sum_x$ and $\sum_y$ to indicate sums that arise over p and s. A subscript x, y or α, β, and ν on the left-hand side of any quantity means that both pairs of longitudinal and transverse forms of the quantity exist and satisfy the relevant equation. Four different equations can be obtained by replacing x and y by either p and/or s.

In order to avoid repetition, we cite key equations of Refs. [3,4,6,14]. In general, we work with spherical obstacles which can be either perfectly elastic, rigid, or fluid-filled cavities. We use bold face or regular vector notation when it is appropriate. The hat on the top of a vector indicates a vector of unit magnitude. The tilde on the top of a letter denotes a dyadic (second-rank tensor). For brevity, we use Eq. ([5]-2) for Eq. (2) of Ref. [5], etc.

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II. GENERAL CONSIDERATIONS

We consider the scattering of an incident plane harmonic elastic wave $\boldsymbol{\phi}$ propagating in a direction $\hat{\mathbf{k}}$ by a doubly periodic planar distribution of identical elastic obstacles. As in Ref. [1], we locate the center of the smallest sphere (of radius a) circumscribing a scatterer at the lattice site $\mathbf{b}_t=\mathbf{b}(t_1,t_2)=t_1b_1\hat{\mathbf{x}}+t_2b_2\hat{\mathbf{y}}$ with $t_i=0,\pm1,\pm2,\ldots$ in the plane $z=0$.

The problem is to determine the total scattered field ${}_x\mathfrak{u}_{\hat{\mathbf{k}}}$ due to the incident wave denoted as an $x=(p,s)$ incident wave and given by ${}_x\boldsymbol{\phi}_{\hat{\mathbf{k}}}$. We suppress the time harmonic dependence $e^{-i\omega t}$ and define the x incident wave as

$$
\begin{aligned}
&{}_p\boldsymbol{\phi}_{\hat{\mathbf{k}}}=\hat{\mathbf{k}}e^{ik_p\hat{\mathbf{k}}\cdot\mathbf{r}},\quad {}_s\boldsymbol{\phi}_{\hat{\mathbf{k}}}=\hat{\boldsymbol{\epsilon}}_s\,e^{ik_s\hat{\mathbf{k}}\cdot\mathbf{r}},\\
&\hat{\mathbf{r}}(\theta;\varphi)=\hat{\boldsymbol{\eta}}(\varphi)\sin\theta+\hat{\mathbf{z}}\cos\theta,\\
&\hat{\boldsymbol{\eta}}(\varphi)=\hat{\mathbf{x}}\cos\varphi+\hat{\mathbf{y}}\sin\varphi,
\end{aligned}
\tag{1}
$$

with $\hat{\mathbf{k}}=\hat{\mathbf{r}}(\theta_0,\varphi_0)=\hat{\mathbf{k}}_0$, $0<\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}\le1$, $\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\epsilon}}_s=0$, and $\hat{\boldsymbol{\epsilon}}_s$ the polarization of the transverse wave.

In the lossless volume external to the scatterers, we work with the total field

$$
{}_x\boldsymbol{\Psi}={}_x\boldsymbol{\phi}_{\hat{\mathbf{k}}}+{}_x\mathfrak{u}_{\mathbf{k}}\ ,
\tag{2}
$$

satisfying (Ref. [15]), in the absence of bodily forces, the time-independent linearized equation of dynamic elasticity

$$
\begin{aligned}
&[c_p^2\nabla(\nabla\cdot)-c_s^2\nabla\times(\nabla\times)+\omega^2]_x\boldsymbol{\Psi}=\mathbf{0}\ ,\\
&\omega=c_pk_p=c_sk_s\ ,\\
&c_p^2=\frac{\lambda+2\mu}{\rho},\quad c_s^2=\frac{\mu}{\rho}\ ,
\end{aligned}
\tag{3}
$$

subject to the usual elastic boundary or transition conditions $\boldsymbol{\psi}=\boldsymbol{\Psi}$, $\mathbf{T}_1\boldsymbol{\psi}=\mathbf{T}_2\boldsymbol{\Psi}$, with $\mathbf{T}_j=2\mu_j\hat{\mathbf{n}}\cdot\nabla+\lambda_j\nabla\cdot+\mu_j\hat{\mathbf{n}}\times\nabla\times$ at the surface of the scatterer. Here λ and μ are the Lamé constants of the embedding medium and ρ is constant density. Similar to Twersky Eq. ([1]-4), the corresponding elastic scattered wave of the doubly periodic planar array may be represented by

$$
\begin{aligned}
&{}_x\mathfrak{u}_{\hat{\mathbf{k}}}=\sum_y {}_x\mathfrak{u}_{\hat{\mathbf{k}}y}=\sum_y\sum_t {}_x\mathbf{U}_y(\mathbf{r}_t)e^{i\mathbf{k}_y\cdot\mathbf{b}_t}\ ,\\
&\sum_t=\sum_{t_1=-\infty}^{\infty}\ \sum_{t_2=-\infty}^{\infty}\ ,
\end{aligned}
\tag{4}
$$

with $\mathbf{r}_t=\mathbf{r}-\mathbf{b}_t$.

The multiple-scattered wave ${}_x\mathbf{U}(\mathbf{r}_t)={}_x\mathbf{U}_p+{}_x\mathbf{U}_s$ of one scatterer is determined as in Ref. [3] by its single elastic-scattering analog and the geometry of the array. In the present context, the multiple-scattered wave ${}_x\mathbf{U}$ will be specified by its corresponding multiple-scattering amplitude ${}_x\mathbf{G}$ which is given in terms of the known scattering amplitudes ${}_x\mathbf{g}_y$ and ${}_x\tilde{\mathbf{g}}_y$ of the single object in isolation.

From Eqs. ([3]-12) and ([3]-13), the single-scattering amplitudes are

$$
{}_x\mathbf{g}_y(\hat{\mathbf{r}},\hat{\mathbf{k}})=(ik_y/4\pi\rho c_y^2)\{\tilde{\mathbf{I}}_ye^{-ik_y\hat{\mathbf{r}}\cdot\mathbf{r}'},{}_x\mathbf{u}(\mathbf{r}')\}\ ,
\tag{5}
$$

where

$$
\tilde{\mathbf{I}}_p=\hat{\mathbf{r}}\hat{\mathbf{r}},\quad \tilde{\mathbf{I}}_s=(\tilde{\mathbf{I}}-\hat{\mathbf{r}}\hat{\mathbf{r}}),\quad \tilde{\mathbf{I}}=\hat{\mathbf{r}}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}+\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}\ ,
\tag{6}
$$

and the brace operator of (5) is defined as in Eq. ([3]-6) by

$$
\{\mathbf{f},\mathbf{g}\}\equiv-(1/4\pi)\int_S[\mathbf{f}\cdot\mathbf{T}_{\mathbf{r}'}\mathbf{g}-\mathbf{g}\cdot\mathbf{T}_{\mathbf{r}'}\mathbf{f}]dS(\mathbf{r}')\ .
\tag{7}
$$

Here

$$
\mathbf{T}_{\mathbf{r}}\equiv2\mu\hat{\mathbf{n}}\cdot\nabla_{\mathbf{r}}+\lambda\hat{\mathbf{n}}\nabla_{\mathbf{r}}\cdot=\mu\hat{\mathbf{n}}\times(\nabla_{\mathbf{r}}\times)
\tag{8}
$$

is the elastic stress tensor, $\hat{\mathbf{n}}$ is the exterior unit normal on the surface S, and $\mathbf{r}$ and $\mathbf{r}'$ denote the observation point and a point on S or in the volume V of the bounded scatterer, respectively. The coefficient in (5), $(ik_y/4\pi\rho c_y^2)=(ik_y^3/4\pi\rho\omega^2)$, gives either the longitudinal or the transverse scale factor needed in the brace operator definition of the single-scattering amplitudes ${}_x\mathbf{g}_y$ and ${}_x\tilde{\mathbf{g}}_y$.

From Eq. ([4]-32) and the single dyadic scattered wave $\tilde{\mathbf{u}}$ specified by $(\tilde{\mathbf{u}}\cdot\hat{\boldsymbol{\epsilon}}=\mathbf{u})$, the single dyadic scattering amplitudes are

$$
\begin{aligned}
&{}_x\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}})=(ik_y/4\pi\rho c_y^2)\{\tilde{\boldsymbol{\phi}}_y(-\hat{\mathbf{r}}),{}_x\tilde{\mathbf{u}}_{\hat{\mathbf{k}}}(\mathbf{r},\hat{\mathbf{k}}')\}\ ,\\
&\tilde{\boldsymbol{\phi}}=\tilde{\boldsymbol{\phi}}_p+\tilde{\boldsymbol{\phi}}_s\equiv\hat{\mathbf{k}}\hat{\mathbf{k}}e^{ik_p\hat{\mathbf{r}}\cdot\mathbf{r}'}+(\tilde{\mathbf{I}}-\hat{\mathbf{k}}\hat{\mathbf{k}})e^{ik_s\hat{\mathbf{r}}\cdot\mathbf{r}'}\ ,
\end{aligned}
\tag{9}
$$

and

$$
{}_x\mathbf{g}_y(\hat{\mathbf{r}},\hat{\mathbf{k}})={}_x\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}})\cdot\hat{\boldsymbol{\epsilon}}_y\ .
\tag{10}
$$

Asymptotically, for $r_t\to\infty$, we can write

$$
{}_x\mathbf{U}(\mathbf{r}_t)\sim\sum_y h(k_y|\mathbf{r}_t|)({}_x\mathbf{G}_y)\ ,
\tag{11}
$$

where

$$
{}_x\mathbf{G}_y={}_x\mathbf{G}_y(\hat{\mathbf{r}}_t;\hat{\mathbf{k}})\ .
\tag{12}
$$

Here $h(k|\mathbf{r}_t|)$ is the spherical Hankel's function of the first kind, and ${}_x\mathbf{G}_p$ and ${}_x\mathbf{G}_s$ are the longitudinal and transverse multiple vector scattering amplitudes due to the incoming x wave and are defined by

$$
\begin{aligned}
{}_x\mathbf{G}_y&={}_x\mathbf{G}_y(\mathbf{r}_t;\hat{\mathbf{r}})\\
&=(ik_y/4\pi\rho c_y^2)\{\tilde{\mathbf{I}}_ye^{-ik_y\hat{\mathbf{r}}\cdot\mathbf{r}'},{}_x\mathbf{U}(\mathbf{r}_t+\mathbf{r}')\}\ .
\end{aligned}
\tag{13}
$$

The spectral representation of the multiple-scattered wave ${}_x\mathbf{U}$ analog to Eq. ([3]-35) is

$$
{}_x\mathbf{U}(\mathbf{r}_t)=\sum_y {}_x\mathbf{U}_y(\mathbf{r}_t)=\sum_y\int_c\{e^{ik_y\hat{\mathbf{r}}_c\cdot\mathbf{r}_t}{}_x\mathbf{G}_{yc}\}\ ,
\tag{14}
$$

where ${}_x\mathbf{G}_{yc}={}_x\mathbf{G}_y(\hat{\mathbf{r}}_c)$, and

$$
\int_c\{\ \}\equiv\frac{1}{2\pi}\int_c\{\ \}d\Omega_c\ .
\tag{15}
$$

The unit vector $\hat{\mathbf{r}}_c$ in (14) is

$$
\hat{\mathbf{r}}_c=\hat{\mathbf{r}}_c(\theta_c,\varphi_c)=\hat{\boldsymbol{\eta}}(\varphi_c)\sin\theta_c+\hat{\mathbf{z}}\cos\theta_c\ ,
\tag{16}
$$

where c is the Sommerfeld's path of Refs. [14,16] and $d\Omega_c$ is the differential solid around the unit vector $\hat{\mathbf{r}}_c$. The integration in (15) is over all angles of observation associated with $\hat{\mathbf{r}}_c$.

Substituting Eq. (11) into (4), and using the method of stationary phase as in Eqs. ([1]-59)–([1]-60), we obtain

$$ {}_x\mathfrak{U}_{\hat{\mathbf{k}}} = \begin{cases} \sum_y \sum_l \mathcal{C}_{yl} e^{ik_y \hat{\mathbf{r}}_{cl}\cdot\mathbf{r}}\, {}_x\mathbf{G}_y(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_0) \quad \text{for } z>a & (17) \\ \sum_y \sum_l \mathcal{C}_{yl} e^{ik_y \hat{\mathbf{r}}'_{cl}\cdot\mathbf{r}}\, {}_x\mathbf{G}_y(\hat{\mathbf{r}}'_{cl},\hat{\mathbf{k}}_0) \quad \text{for } z<a\,. & (18) \end{cases} $$

Here

$$ \mathcal{C}_{yl} = \frac{2\pi}{k_y^2 b_1 b_2 \cos\theta_{cl}}\,, \quad \sum_l = \sum_{l_1}\sum_{l_2}\,, \quad l_i = 0, \pm 1, \pm 2, \ldots, \tag{19} $$

and $\hat{\mathbf{r}}'_{cl} = \hat{\mathbf{r}}_{cl}(-\hat{\mathbf{z}})$. For the treatment of the limiting case with $\mathcal{C}_{yl}\to\infty$ and the study of resonance for moderate values of the parameter kb_i, we refer the reader to the work of Twersky in Refs. [17–20,22].

From Eqs. ([3]-46), ([3]-47) adapted to the doubly elastic periodic array, and Eq. ([1]-12), we obtain the mixed vector-dyadic representation of the unknown multiple-scattering amplitudes ${}_x\mathbf{G}_y$ of Eq. (11) in terms of the known scattering amplitudes ${}_x\mathbf{g}_y$ or ${}_x\tilde{\mathbf{g}}_y$ of the single scatterer in isolation,

$$ {}_x\mathbf{G}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) = {}_x\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\cdot\hat{\boldsymbol{\epsilon}}_y + \sum_\alpha \sum_t{}' e^{i\mathbf{k}_\alpha\cdot\mathbf{b}_t} \int_c \{ e^{-i\mathbf{k}_{ac}\cdot\mathbf{b}_t} [{}_\alpha\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{r}}_c)]\cdot {}_x\mathbf{G}_\alpha(\hat{\mathbf{r}}_c,\hat{\mathbf{k}}_0)\}\,, \tag{20} $$

where $\mathbf{k}_\alpha = k_\alpha \hat{\mathbf{k}}_0$, $\mathbf{k}_{ac} = k_\alpha \hat{\mathbf{r}}_c$, $\hat{\boldsymbol{\epsilon}}_p \| {}_p\hat{\mathbf{g}}_p$, $\hat{\boldsymbol{\epsilon}}_s \| {}_s\hat{\mathbf{g}}_s$, and $\sum'_t = \sum_{t_1\neq 0}\sum_{t_2\neq 0}$. Equations (17)–(20) determine the multiple vector scattering amplitudes of one scatterer in terms of its response and these of its neighbors to an incoming incident elastic wave ${}_x\boldsymbol{\phi}_{\hat{\mathbf{k}}}$.

III. ENERGY CONSIDERATIONS

In this section, we restrict discussion only to lossless scatterers and follow the procedure of Phanord-Berger of Ref. [6], Sec. 5. We establish elastic energy theorems for multiple-scattering amplitudes separately, multiple- and single-scattering amplitudes together, and single-scattering amplitudes alone. Because of the four different multiple reciprocity relations of Ref. [6], we expect four different energy theorems from each group mentioned above. To provide continuity of thoughts and facilitate the understanding of the present work, we sketch briefly the derivations for p-p and p-s incident waves based on Ref. [6], and use them to obtain energy theorems for s-p and s-s cases.

In the context of elastic scattering, when the concept of "energy" is mentioned, it is understood to be the "total energy" as the incident wave interacts with the scatterer or scatterers in the medium of propagation. The effect or response of the scatterers on the propagation of the incident wave is measured, as in Refs. [14–20], by the amount of energy that they receive from the incident wave as it gets reradiated in all directions. But, as mentioned in Sec. II, the multiple-scattered wave ${}_x\mathbf{U}$, which is the response of the scatterers to the incident wave, is specified by its corresponding multiple-scattering amplitude ${}_x\mathbf{G}$. Therefore, the energy theorems of this section will be expressed or given in terms of the scattering amplitudes.

From general reciprocity of Ref. [5], for two arbitrary longitudinal directions of incidence $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$, we have

$$ \{ {}_p\boldsymbol{\Psi}^*_{\hat{\mathbf{k}}_1}, {}_p\boldsymbol{\Psi}_{\hat{\mathbf{k}}_2} \}_m = \{ {}_p\boldsymbol{\Psi}^*_1, {}_p\boldsymbol{\Psi}_2 \}_m = 0\,, \tag{21} $$

where m is the surface of the central scatterer and ${}_p\boldsymbol{\Psi}_i$ is the total outside multiple field due to ${}_p\boldsymbol{\phi}_i$. Here, the asterisk denotes complex conjugation and the brace operator $\{\mathbf{f}_1,\mathbf{f}_2\}_m$ of (21) is the regular Betti's surface integral of dynamic elasticity of Ref. [15] defined by Eq. (7). Substituting (2) into (21) and applying brace algebra of Eq. ([3]-39) lead to

$$ \{ {}_p\boldsymbol{\phi}^*_1, {}_p\mathfrak{U}_2 \}_m - \{ {}_p\boldsymbol{\phi}^*_2, {}_p\mathfrak{U}_1 \}^*_m = -\{ {}_p\mathfrak{U}^*_1, {}_p\mathfrak{U}_2 \}_m\,, \tag{22} $$

where $\{ {}_p\boldsymbol{\phi}^*_1, {}_p\boldsymbol{\phi}_2\}_m = 0$ since ${}_p\boldsymbol{\phi}_1$ and ${}_p\boldsymbol{\phi}_2$ are two different nonsingular solutions of (3).

Using Eqs. ([6]-47) and ([6]-52), we reduce (22) to

$$ \beta_p[\hat{\mathbf{k}}_1\cdot {}_p\mathbf{G}_p(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2) + \hat{\mathbf{k}}_2\cdot {}_p\mathbf{G}^*_p(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)] = \{ {}_p\mathfrak{U}^*_1, {}_p\mathfrak{U}_2\}_m\,. \tag{23} $$

The right-hand side of (23), with (17) and (18), give

$$ \{ {}_p\mathfrak{U}^*_1, {}_p\mathfrak{U}_2\}_m = -\sum_x\sum_l \mathcal{C}_{xl}\{ [e^{ik_x\hat{\mathbf{r}}_{cl}\cdot\mathbf{r}}\, {}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1) + e^{ik_x\hat{\mathbf{r}}'_{cl}\cdot\mathbf{r}}\, {}_p\mathbf{G}_x(\hat{\mathbf{r}}'_{cl},\hat{\mathbf{k}}_1)]^*, {}_p\mathfrak{U}_2\}_m\,. \tag{24} $$

Applying Eqs. ([6]-33) and ([6]-48) to (24) yields

$$ \{ {}_p\mathfrak{U}^*_1, {}_p\mathfrak{U}_2\}_m = -\sum_x \beta_x \sum_l \mathcal{C}_{xl}[{}_p\mathbf{G}^*_x(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot {}_p\mathbf{G}_x(\hat{\mathbf{r}}^*_{cl},\hat{\mathbf{k}}_2) + {}_p\mathbf{G}^*_x(\hat{\mathbf{r}}'_{cl},\hat{\mathbf{k}}_1)\cdot {}_p\mathbf{G}_x(\hat{\mathbf{r}}^{*\prime}_{cl},\hat{\mathbf{k}}_2)]\,, \tag{25} $$

where $\beta_x = [(4\pi i c_x^2/k_x)\rho]$ and $\sum_l = \sum_{l_1}\sum_{l_2}$. Combining Eq. (22) with Eqs. (23) and (25) leads to the multiple p-p energy theorem for the doubly periodic planar array.

$$-\beta_p[\hat{\mathbf{k}}_1\cdot{}_p\mathbf{G}_p(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\hat{\mathbf{k}}_2\cdot{}_p\mathbf{G}_p^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=\sum_x\beta_x\sum_l \mathcal{C}_{xl}[{}_p\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_p\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] . \quad (26)$$

In the forward direction, assuming that the directions of longitudinal and transverse waves coincide, we obtain the multiple *p-p* "forward energy theorem"

$$-2\beta_p\mathrm{Re}[\hat{\mathbf{k}}\cdot{}_p\mathbf{G}_p(\hat{\mathbf{k}},\hat{\mathbf{k}})]=\sum_x\beta_x\sum_l \mathcal{C}_{xl}[|{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}})|^2+|{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}})|^2] , \quad (27)$$

where Re denotes the real part. From (26), we deduce the multiple *s-s* "energy theorem"

$$-\beta_s[\hat{\boldsymbol{\epsilon}}_{s1}\cdot{}_s\mathbf{G}_s(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\hat{\boldsymbol{\epsilon}}_{s2}\cdot{}_s\mathbf{G}_s^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=\sum_x\beta_x\sum_l \mathcal{C}_{xl}[{}_s\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_s\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] , \quad (28)$$

which in the forward direction becomes

$$-2\beta_s\,\mathrm{Re}[\hat{\boldsymbol{\epsilon}}_s\cdot{}_s\mathbf{G}_s(\hat{\mathbf{k}},\hat{\mathbf{k}})]=\sum_x\beta_x\sum_l \mathcal{C}_{xl}[|{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}})|^2+|{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}})|^2] . \quad (29)$$

For *p-s* incident waves, we recast (22) into

$$\{{}_p\boldsymbol{\phi}_1^*,{}_s\mathfrak{U}_2\}_m-\{{}_s\boldsymbol{\phi}_2^*,{}_p\mathfrak{U}_1\}_m^*=-\{{}_p\mathfrak{U}_1^*,{}_s\mathfrak{U}_2\}_m . \quad (30)$$

Therefore (23) gives

$$[\beta_p\hat{\mathbf{k}}_1\cdot{}_s\mathbf{G}_p(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\beta_s\hat{\boldsymbol{\epsilon}}_{s2}\cdot{}_p\mathbf{G}_s^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=\{{}_p\mathfrak{U}_1^*,{}_s\mathfrak{U}_2\}_{\mathrm{m}} , \quad (31)$$

and (25) yields

$$\{{}_p\mathfrak{U}_1^*,{}_s\mathfrak{U}_2\}_{\mathrm{m}}=-\sum_x\beta_x\sum_l \mathcal{C}_{xl}[{}_p\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_p\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] . \quad (32)$$

Combining (31) and (32) leads to the multiple *p-s* energy theorem

$$-[\beta_p\hat{\mathbf{k}}_1\cdot{}_s\mathbf{G}_p(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\beta_s\hat{\boldsymbol{\epsilon}}_{s2}\cdot{}_p\mathbf{G}_s^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=\sum_x\beta_x\sum_l \mathcal{C}_{xl}[{}_p\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_p\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] . \quad (33)$$

Hence the multiple *s-p* energy theorem obtained from (33) is

$$-[\beta_s\hat{\boldsymbol{\epsilon}}_{s1}\cdot{}_p\mathbf{G}_s(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\beta_p\hat{\mathbf{k}}_2\cdot{}_s\mathbf{G}_p^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=\sum_x\beta_x\sum_l \mathcal{C}_{xl}[{}_s\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_s\mathbf{G}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] . \quad (34)$$

In the forward direction, (33) and (34) are identically zero since all mixed vector scattering amplitudes must vanish (Ref. [21], pp. 1610, 1748–1750). Equations (26), (28), (33), and (34) are the four different multiple-scattering energy theorems for the doubly periodic planar array of elastic scatterers. They relate the multiple vector scattering amplitudes or the multiple-scattered fields for two arbitrary directions of incidence at the surface of the central scatterer. Equations (26) and (28) are the elastic analogs of Eqs. ([1]-18) and ([22]-32), respectively. Equations (27) and (29) correspond to the usual theorems on conservation of energy in the forward direction of scattering.

Now, consider

$$\{{}_p\boldsymbol{\psi}_1^*,{}_p\boldsymbol{\Psi}_2\}_m=0 , \quad (35)$$

where ${}_p\boldsymbol{\psi}_i={}_p\boldsymbol{\phi}_i+{}_p\mathfrak{u}_i$ is the total outside single solution for the central scatterer located at m due to ${}_p\boldsymbol{\phi}_i$. Working with Eq. (35), we obtain, as in (22),

$$\{{}_p\boldsymbol{\phi}_1^*,{}_p\mathfrak{U}_2\}_m-\{{}_p\boldsymbol{\phi}_2^*,{}_p\mathfrak{u}_1\}_m^*=-\{{}_p\mathfrak{u}_1^*,{}_p\mathfrak{U}_2\}_m . \quad (36)$$

Similarly to (23), Eq. (36) gives

$$\beta_p[\hat{\mathbf{k}}_1\cdot{}_p\mathbf{G}_p(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\hat{\mathbf{k}}_2\cdot{}_p\mathbf{g}_p^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=\{{}_p\mathfrak{u}_1^*,{}_p\mathfrak{U}_2\}_{\mathrm{m}} . \quad (37)$$

In the right-hand side of (37), we use the spectral representation Eq. ([3]-19) of ${}_p\mathfrak{u}_1$, complex conjugation, and Eq. ([6]-48) to write

$$\{{}_p\mathfrak{u}_1^*,{}_p\mathfrak{U}_2\}_{\mathrm{m}}=-\sum_x\beta_x\int_c^*\{{}_p\mathbf{G}_x(\hat{\mathbf{r}}_c^*,\hat{\mathbf{k}}_2)\cdot{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_c,\hat{\mathbf{k}}_1)+{}_p\mathbf{G}_x(\hat{\mathbf{r}}_c^{*\prime},\hat{\mathbf{k}}_2)\cdot{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_c',\hat{\mathbf{k}}_1)\} . \quad (38)$$

Working similarly as in Eq. (17) and (18), and introducing as in Eq. ([1]-7) or ([5]-10) the continuous variables l_1 and l_2, we transform (38) into

$$\{{}_p\mathfrak{u}_1^*,{}_p\mathfrak{U}_2\}_{\mathrm{m}}=-\sum_x\beta_x\int_l \mathcal{C}_{xl}[{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)\cdot{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_c,\hat{\mathbf{k}}_1)+{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)\cdot{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_c',\hat{\mathbf{k}}_1)] , \quad (39)$$

where $\int_l=\int dl_1\int dl_2$ and the limits on (l_1,l_2) as in Ref. [1] are over the propagating range of l which depends on Eq. (16) or more precisely on $\cos\theta_{cl}=[1-\sin^2(\theta_{cl})]^{1/2}$ with $\sin^2(\theta_{cl})<1$.

Combining (37) with (39) leads to the mixed *p-p* energy theorem

$$\beta_p[\hat{\mathbf{k}}_1\cdot{}_p\mathbf{G}_p(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\hat{\mathbf{k}}_2\cdot{}_p\mathbf{g}_p^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] \;. \tag{40}$$

For *p-s* incident waves, we use (37) and (39) to obtain

$$\beta_p\hat{\mathbf{k}}_1\cdot{}_s\mathbf{G}_p(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\beta_s\hat{\boldsymbol{\epsilon}}_{s2}\cdot{}_p\mathbf{g}_s^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)=\{{}_p\mathbf{u}_1^*,{}_s\mathbf{u}_2\}_{\mathrm{m}} \;, \tag{41}$$

and

$$\{{}_p\mathbf{u}_1^*,{}_s\mathbf{u}_2\}_{\mathrm{m}}=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)\cdot{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)+{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)\cdot{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)] \;. \tag{42}$$

Hence Eqs. (41) and (42) give the mixed *p-s* energy theorem

$$\beta_p\hat{\mathbf{k}}_1\cdot{}_s\mathbf{G}_p(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\beta_s\hat{\boldsymbol{\epsilon}}_{s2}\cdot{}_p\mathbf{g}_s^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] \;. \tag{43}$$

The remaining two cases corresponding to *s-s* and *s-p* incident waves are deduced from (40) and (43). Therefore, the mixed *s-s* energy theorem is

$$\beta_s[\hat{\boldsymbol{\epsilon}}_{s1}\cdot{}_s\mathbf{G}_s(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\hat{\boldsymbol{\epsilon}}_{s2}\cdot{}_s\mathbf{g}_s^*(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_1)]=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_s\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_s\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] \;, \tag{44}$$

and for the mixed *s-p* energy theorem, we have

$$\beta_s\hat{\boldsymbol{\epsilon}}_{s1}\cdot{}_p\mathbf{G}_s(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\beta_p\hat{\mathbf{k}}_2\cdot{}_s\mathbf{g}_p^*(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_1)=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_s\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_s\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{G}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] \;. \tag{45}$$

Equations (40) and (43)–(45) are energy-conservation requirements involving the total multiple field of the doubly periodic planar array and the net outside field of an individual scatterer at its surface.

In Eq. (35), we replace ${}_p\boldsymbol{\Psi}_2$ by ${}_p\boldsymbol{\psi}_2={}_p\boldsymbol{\phi}_2+{}_p\mathbf{u}_2$ and rewrite (36) as

$$\{{}_p\boldsymbol{\phi}_1^*,{}_p\mathbf{u}_2\}_m-\{{}_p\boldsymbol{\phi}_2^*,{}_p\mathbf{u}_1\}_m^*=-\{{}_p\mathbf{u}_1^*,{}_p\mathbf{u}_2\}_m \;. \tag{46}$$

Proceeding similarly to (37) and (39), the *p-p* energy theorem for a single scatterer in the array is

$$\beta_p[\hat{\mathbf{k}}_1\cdot{}_p\mathbf{g}_p(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)+\hat{\mathbf{k}}_2\cdot{}_p\mathbf{g}_p^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] \;, \tag{47}$$

which, in forward direction, gives

$$-2\beta_p\,\mathrm{Re}[\hat{\mathbf{k}}\cdot{}_p\mathbf{g}_p(\hat{\mathbf{k}},\hat{\mathbf{k}})]=\sum_x\beta_x\int_l\mathcal{C}_{xl}[|{}_p\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}})|^2+|{}_p\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}})|^2] \;. \tag{48}$$

The *s-s* energy theorem for single-scattering amplitudes analog to (47) is

$$\beta_s[\hat{\boldsymbol{\epsilon}}_{s1}\cdot{}_s\mathbf{g}_s(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)+\hat{\boldsymbol{\epsilon}}_{s2}\cdot{}_s\mathbf{g}_s^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)]=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_s\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_s\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] \;, \tag{49}$$

which, in the forward direction, becomes as in (48),

$$-2\beta_s\,\mathrm{Re}[\hat{\boldsymbol{\epsilon}}_s\cdot{}_s\mathbf{g}_s(\hat{\mathbf{k}},\hat{\mathbf{k}})]=\sum_x\beta_x\int_l\mathcal{C}_{xl}[|{}_s\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}})|^2+|{}_s\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}})|^2] \;. \tag{50}$$

For *p-s* incident waves, we use Eq. (30) and (46) to write

$$\{{}_p\boldsymbol{\phi}_1^*,{}_s\mathbf{u}_2\}_m-\{{}_s\boldsymbol{\phi}_2^*,{}_p\mathbf{u}_1\}_m^*=-\{{}_p\mathbf{u}_1^*,{}_s\mathbf{u}_2\}_m \;. \tag{51}$$

Following Eq. (43) and using (51) yields the *p-s* energy theorem for the isolated scatterer,

$$\beta_p\hat{\mathbf{k}}_1\cdot{}_s\mathbf{g}_s(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\beta_s\hat{\boldsymbol{\epsilon}}_{2s}\cdot{}_p\mathbf{g}_s^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_p\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_s\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] \;. \tag{52}$$

Finally, we obtain from (52) the *s-p* energy theorem for the individual scatterer in the doubly periodic array,

$$\beta_s\hat{\boldsymbol{\epsilon}}_{s1}\cdot{}_p\mathbf{g}_s(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2)+\beta_p\hat{\mathbf{k}}_2\cdot{}_s\mathbf{g}_p^*(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_1)=-\sum_x\beta_x\int_l\mathcal{C}_{xl}[{}_s\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl},\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}}_2)+{}_s\mathbf{g}_x^*(\hat{\mathbf{r}}_{cl}',\hat{\mathbf{k}}_1)\cdot{}_p\mathbf{g}_x(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}}_2)] \;. \tag{53}$$

Equations (47), (49), (52), and (53) relate the single vector scattering amplitudes or the single-scattered fields for two arbitrary directions of incidence at the surface of the central scatterer. They can be further specialized by using either Dassios's reciprocity relations of Ref. [23] or Varatharajulu's of Ref. [24] for single-scattering amplitudes to replace ${}_x\mathbf{g}_y^*$.

IV. LEADING TERM APPROXIMATION AND RAYLEIGH SCATTERING FOR SMALL ELASTIC RIGID SCATTERERS

The purpose of this section is to recast the complicated theorems of Sec. III into forms ready for applications. To accomplish this goal, we will employ the leading term approximation used by Twersky in Refs. [1,4,25] for the multiple- and single-scattering amplitudes. The multiple-scattering amplitudes represent the response at large distance (the radiation zone) of the doubly periodic planar array of elastic obstacles to the incident wave. The leading term approximation is most valid in the radiation or far zone and it is also known as the Born approximation (for more details, see Ref. [26]). As an application of the simplified forms of the theorems, we will look at Rayleigh scattering for elastic rigid scatterers since their size parameter ka is small as compared to wavelength (Ref. [26]).

To simplify the notation and facilitate subsequent developments, we introduce as in Eqs. ([4]-78) and ([25]-128) the dyadic multiple-scattering amplitude ${}_x\tilde{\mathbf{G}}_y$ such that ${}_x\mathbf{G}_y={}_x\tilde{\mathbf{G}}_y\cdot\hat{\boldsymbol{\epsilon}}_y$, and we transform Eq. (20) into a compact form by dropping the $\hat{\boldsymbol{\epsilon}}_y$,

$$ {}_x\tilde{\mathbf{G}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)={}_x\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)+\sum_\alpha\sum_t{}' e^{i\mathbf{k}_\alpha\cdot\mathbf{b}_t}\int_c\{e^{-i\mathbf{k}_{ac}\cdot\mathbf{b}_t}[{}_\alpha\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{r}}_c)]\cdot{}_x\tilde{\mathbf{G}}_\alpha(\hat{\mathbf{r}}_c,\hat{\mathbf{k}}_0)\}\ . \tag{54} $$

Now, using Eq. ([4]-88) or ([25]-55) and keeping $k_y b_t$ large, we reduce Eq. (54) to

$$ \begin{aligned} {}_x\tilde{\mathbf{G}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)&={}_x\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\\ &\quad+\sum_\alpha\sum_t{}'\,{}_\alpha\tilde{\mathfrak{F}}_t\cdot{}_\alpha\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{b}}_t)\cdot{}_x\tilde{\mathbf{G}}_\alpha(\hat{\mathbf{b}}_t,\hat{\mathbf{k}}_0)\ ,\\ {}_\alpha\tilde{\mathfrak{F}}_t&\equiv h(tk_\alpha b_t)e^{-i\mathbf{k}_\alpha\cdot\mathbf{b}_t}\tilde{\mathfrak{D}}_\alpha(tk_\alpha b_t;\tilde{\mathcal{D}})\ , \end{aligned} \tag{55} $$

in which

$$ \tilde{\mathfrak{D}}_\alpha(tk_\alpha b_t l;\tilde{\mathcal{D}})\equiv\tilde{I}+\left[\frac{i}{2tk_\alpha b_t}\tilde{\mathcal{D}}\right]+\left[\frac{i}{2tk_\alpha b_t}\right]^2\left[\frac{1}{2}\right]\tilde{\mathcal{D}}\cdot(\tilde{\mathcal{D}}-1\cdot2\tilde{\mathbf{I}})+\cdots \tag{56} $$

is the dyadic operator of Eq. ([25]-50) with $\tilde{\mathcal{D}}\equiv r^2\{\nabla\times\nabla-\nabla\nabla\cdot\}$. The spherical-polar coordinates representation of $\tilde{\mathcal{D}}$ is given by Eq. ([25]-44), and the differentiations are with respect to the angles associated to the unit vector $\hat{\mathbf{b}}_t$.

For application purposes, we approximate all operators of (55) by their leading terms and retain only contributions up to the second order of scattering. Consequently, (55) is transformed into

$$ \begin{aligned} {}_x\tilde{\mathbf{G}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)&\simeq{}_x\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)+\sum_\alpha\sum_t{}'\,\tilde{\mathcal{H}}_{\alpha t}\cdot{}_\alpha\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{b}}_t)\cdot{}_x\tilde{\mathbf{g}}_\alpha(\hat{\mathbf{b}}_t,\hat{\mathbf{k}}_0)\\ \tilde{\mathcal{H}}_{\alpha t}&\equiv h(tk_\alpha b_t)e^{-i\mathbf{k}_\alpha\cdot\mathbf{b}_t}\tilde{\mathbf{I}}\ , \end{aligned} \tag{57} $$

which becomes for $\hat{\mathbf{r}}=\hat{\mathbf{b}}_t$, dot-multiplication from the right by $\hat{\boldsymbol{\epsilon}}_y$, and $\alpha=y$,

$$ \begin{aligned} {}_x\mathbf{G}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)&\simeq\tilde{\mathcal{R}}_{yt}\cdot{}_x\mathbf{g}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\cdot\hat{\boldsymbol{\epsilon}}_y\ ,\\ \tilde{\mathcal{R}}_{yt}&\equiv\{\tilde{\mathbf{I}}+\sum_t{}'\,\tilde{\mathcal{H}}_{yt}\cdot{}_y\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{r}})\}\ , \end{aligned} \tag{58} $$

since ${}_p\tilde{\mathbf{g}}_s(\hat{\mathbf{r}},\hat{\mathbf{r}})={}_s\tilde{\mathbf{g}}_p(\hat{\mathbf{r}},\hat{\mathbf{r}})=\tilde{0}$ (Ref. [21], pp. 1610, 1748, and 1750). When $x=p$ or s and $y=p$, we obtain from (58) the longitudinal forms

$$ \begin{aligned} {}_p\mathbf{G}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)&\simeq\tilde{\mathcal{R}}_{pt}\cdot{}_p\tilde{\mathbf{g}}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\cdot\hat{\boldsymbol{\epsilon}}_p\ ,\\ {}_s\mathbf{G}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)&\simeq\tilde{\mathcal{R}}_{pt}\cdot{}_s\tilde{\mathbf{g}}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\cdot\hat{\boldsymbol{\epsilon}}_p\ , \end{aligned} \tag{59} $$

respectively. For $x=s$ or p and $y=s$, we have the transverse analog of (59),

$$ \begin{aligned} {}_s\mathbf{G}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)&\simeq\tilde{\mathcal{R}}_{st}\cdot{}_s\tilde{\mathbf{g}}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\cdot\hat{\boldsymbol{\epsilon}}_s\ ,\\ {}_p\mathbf{G}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)&\simeq\tilde{\mathcal{R}}_{st}\cdot{}_p\tilde{\mathbf{g}}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\cdot\hat{\boldsymbol{\epsilon}}_s\ . \end{aligned} \tag{60} $$

To obtain specific results, we let the identical scatterers by rigid spheres of radius a and rewrite the single dyadic scattering amplitudes ${}_p\tilde{\mathbf{g}}_p(\hat{\mathbf{r}},\hat{\mathbf{r}})$ and ${}_s\tilde{\mathbf{g}}_s(\hat{\mathbf{r}},\hat{\mathbf{r}})$ in the forward direction of scattering as

$$ {}_p\tilde{\mathbf{g}}_p(\hat{\mathbf{r}},\hat{\mathbf{r}})={}_pg_p(\hat{\mathbf{r}},\hat{\mathbf{r}})\hat{\mathbf{r}}\hat{\mathbf{r}},\quad {}_s\tilde{\mathbf{g}}_s(\hat{\mathbf{r}},\hat{\mathbf{r}})={}_sg_s(\hat{\mathbf{r}},\hat{\mathbf{r}})\tilde{\mathbf{I}}_*\ , \tag{61} $$

where, as in Eq. ([3]-14) or Eq. (6), $\tilde{\mathbf{I}}=\hat{\mathbf{r}}\hat{\mathbf{r}}+\tilde{\mathbf{I}}_*$, and $\tilde{\mathbf{I}}_*\equiv\tilde{\mathbf{I}}-\hat{\mathbf{r}}\hat{\mathbf{r}}=\hat{\theta}\hat{\theta}+\hat{\varphi}\hat{\varphi}$. Next, we use the second part of (57) to reduce the second part of (58) to

$$ \begin{aligned} \tilde{\mathcal{R}}_{yt}&\equiv\tilde{\mathbf{I}}+\sum_t{}'h(tk_yb_t)e^{-ik_yb_t\hat{\mathbf{k}}_0\cdot\hat{\mathbf{r}}}[{}_y\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{r}})]\\ &=\tilde{\mathbf{I}}+\overline{\mathcal{H}}_y(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})\,{}_y\tilde{\mathbf{g}}_y(\hat{\mathbf{r}},\hat{\mathbf{r}})\ , \end{aligned} \tag{62} $$

where the scalar factor

$$ \begin{aligned} \overline{\mathcal{H}}_y(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})&\equiv\sum_t{}'h(tk_yb_t)e^{-ik_yb_t\hat{\mathbf{k}}_0\cdot\hat{\mathbf{r}}}\\ &=(ik_y)^{-1}\sum_t{}'\frac{e^{itk_yb_{t_i}(1-\hat{\mathbf{k}}_0\cdot\hat{\mathbf{r}})}}{tb_t}\ , \end{aligned} \tag{63} $$

which is convergent Eqs. ([5]-39), ([5]-40) except for the integral values of $\mathcal{V}_{t_i}=(k_yb_{t_i}/2\pi)(1-\hat{\mathbf{k}}_0\cdot\hat{\mathbf{r}})$.

Combining (62) with (61) leads to

$$\begin{aligned}\tilde{\mathcal{R}}_{pt} &= \tilde{\mathbf{I}} + \{\overline{\mathcal{H}}_p(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})_p g_p(\hat{\mathbf{r}},\hat{\mathbf{r}})\}\hat{\mathbf{r}}\hat{\mathbf{r}} ,\\ \tilde{\mathcal{R}}_{st} &= \{1+\overline{\mathcal{H}}_s(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})_s g_s(\hat{\mathbf{r}},\hat{\mathbf{r}})\}\tilde{\mathbf{I}}\\ &\quad -\{\overline{\mathcal{H}}_s(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})_s g_s(\hat{\mathbf{r}},\hat{\mathbf{r}})\}\hat{\mathbf{r}}\hat{\mathbf{r}} .\end{aligned} \tag{64}$$

Using the results of Ref. [14], Sec. 2 for the forward scalar single-scattering amplitudes $_p g_p(\hat{\mathbf{r}},\hat{\mathbf{r}})$ and $_s g_s(\hat{\mathbf{r}},\hat{\mathbf{r}})$ with small $k_y a$ in (64) yields

$$\begin{aligned}\tilde{\mathcal{R}}_{pt} = \tilde{\mathbf{I}} + \Bigg\{ &\overline{\mathcal{H}}_p(\hat{\mathbf{k}}_0,\hat{\mathbf{r}}) \left[\frac{3}{2}a\right]\left[\frac{k_p}{k_s}\right]^2\\ &+\left[1+\frac{k_p^2}{2k_s^2}\right]^{-1}\left[1+ik_s a\left[1+\frac{k_p^3}{2k_s^3}\right]\right.\\ &\left.\times\left[1+\frac{k_p^2}{2k_s^2}\right]^{-1}\right]\Bigg\}\hat{\mathbf{r}}\hat{\mathbf{r}} ,\end{aligned} \tag{65}$$

and

$$\begin{aligned}\tilde{\mathcal{R}}_{st} = \tilde{\mathbf{I}} + \overline{\mathcal{H}}_s(\hat{\mathbf{k}}_0,\hat{\mathbf{r}}) &\left[\frac{3}{2}a\right]\left[1+\frac{k_p^2}{2k_s^2}\right]^{-1}\\ &\times\left[1-ik_s a\left[1+\frac{k_p^3}{2k_s^3}\right]\right.\\ &\left.\times\left[1+\frac{k_p^2}{2k_s^2}\right]^{-1}\right]\{\tilde{\mathbf{I}}-\hat{\mathbf{r}}\hat{\mathbf{r}}\} .\end{aligned} \tag{66}$$

Substituting Eq. (65) into (59) gives, for the longitudinal forms,

$$\begin{aligned}{}_p\mathbf{G}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) &\simeq \{{}_p\mathcal{F}(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})\}_p g_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)_p\hat{\mathbf{g}}_p ,\\ {}_p\mathcal{F}(\hat{\mathbf{k}}_0,\hat{\mathbf{r}}) &\equiv 1+\overline{\mathcal{H}}_p(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})\left[\frac{3}{2}a\right]\left[\frac{k_p}{k_s}\right]^2\left[1+\frac{k_p^2}{2k_s^2}\right]^{-1}\\ &\quad\times\left[1+ik_s a\left[1+\frac{k_p^3}{2k_s^3}\right]\left[1+\frac{k_p^2}{2k_s^2}\right]^{-1}\right] ,\end{aligned} \tag{67}$$

and

$${}_s\mathbf{G}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) \simeq \{{}_p\mathcal{F}(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})\}_s g_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)_s\hat{\mathbf{g}}_p . \tag{68}$$

For the transverse forms, we combine Eq. (60) with (66) to obtain

$$\begin{aligned}{}_s\mathbf{G}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) &\simeq \{{}_s\mathcal{F}(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})\}_s g_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)_s\hat{\mathbf{g}}_s ,\\ {}_s\mathcal{F}(\hat{\mathbf{k}}_0,\hat{\mathbf{r}}) &\equiv 1+\overline{\mathcal{H}}_s(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})\left[\frac{3}{2}a\right]\left[1+\frac{k_p^2}{2k_s^2}\right]^{-1}\\ &\quad\times\left[1-ik_s a\left[1+\frac{k_p^3}{2k_s^3}\right]\left[1+\frac{k_p^2}{2k_s^2}\right]^{-1}\right] ,\end{aligned} \tag{69}$$

and

$${}_p\mathbf{G}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) \simeq \{{}_s\mathcal{F}(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})\}_p g_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)_p\hat{\mathbf{g}}_s . \tag{70}$$

Using Ref. [27], Appendix B and Ref. [28] in Eqs. (67)–(70) yields the desired Rayleigh's approximation of the four-vector multiple-scattering amplitudes $_x\mathbf{G}_y(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)$ when $x=p,s$, and $y=p,s$ for the identical rigid elastic spheres. Hence working with (67) and (69) we have

$$\begin{aligned}{}_p\mathbf{G}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) &\simeq \sum_{n=0}^{\infty}\mathcal{A}_n\frac{k_p}{i^{n-1}}\{P_n(\cos\theta)\}_p\mathcal{F}(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\hat{\mathbf{r}} ,\\ {}_s\mathbf{G}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) &\simeq \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\Bigg\{\mathcal{B}_n\left[\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}P_n^1(\cos\theta)\cos\varphi-\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta}\{P_n^1(\cos\theta)\}\sin\varphi\right]\\ &\quad+\mathcal{C}_n\left[\hat{\boldsymbol{\theta}}\cos\varphi\frac{\partial}{\partial\theta}\{P_n^1(\cos\theta)\}-\frac{\hat{\boldsymbol{\varphi}}}{\sin\theta}\sin\varphi\{P_n^1(\cos\theta)\}\right]\Bigg\}_s\mathcal{F}(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) ,\end{aligned} \tag{71}$$

and with (68) and (70) we obtain

$$\begin{aligned}{}_p\mathbf{G}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) &\simeq \sum_{n=1}^{\infty}\mathbb{A}_n\frac{k_p}{i^{n-1}}\frac{\partial}{\partial\theta}\{P_n(\cos\theta)\}_s\mathcal{F}(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\hat{\boldsymbol{\theta}} ,\\ {}_s\mathbf{G}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0) &\simeq \sum_{n=1}^{\infty}\mathbb{B}_n\frac{2n+1}{n(n+1)}ik_p\{P_n^1(\cos\theta)\cos\varphi\}_p\mathcal{F}(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)\hat{\mathbf{r}} ,\end{aligned} \tag{72}$$

where $\mathcal{A}_n$ and $\mathbb{A}_n$ are given by Ref. [27], Appendix B, Eq. (9). The remaining scattering coefficients $\mathcal{B}_n$, $\mathcal{C}_n$, and $\mathbb{B}_n$ can be found in Ref. [28].

The results of (71) and (72) for the small rigid elastic spheres are modulated as in Ref. [2] by the geometry of the doubly periodic planar array and contain the symmetry preserving factor $\mathcal{H}_y(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})$ with $y=p,s$. They also show that the leading contribution of the Rayleigh's approximation for $_p\mathbf{G}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)$ is a monopole term while $_s\mathbf{G}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)$, $_p\mathbf{G}_s(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)$, and $_s\mathbf{G}_p(\hat{\mathbf{r}},\hat{\mathbf{k}}_0)$ have a dipole term.

Equations (71) and (72) can be used to obtain the Rayleigh's approximation of the energy theorems of Sec. III. As an example, we look at the multiple energy theorems in the forward direction of scattering of Eq. (27). Keeping only the first nonvanishing terms of the multiple vector scattering amplitudes of Eqs. (71) and (72), we obtain from (27)

$$2\beta_p \,\mathrm{Re}\{\mathcal{A}_0(\hat{\mathbf{r}}) i k_p [{}_p\mathcal{F}(\hat{\mathbf{k}},\hat{\mathbf{k}})]\} = \beta_p \sum_l \mathcal{C}_{pl} k_p^2 [\,||\mathcal{A}_0(\hat{\mathbf{r}}_{cl}^*)|^2 {}_p\mathcal{F}(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}})|^2 + ||\mathcal{A}_0(\hat{\mathbf{r}}_{cl}^{*\prime})|^2 {}_p\mathcal{F}(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}})|^2]$$

$$-\beta_s \sum_l \mathcal{C}_{sl} (k_p \sin\theta)^2 [\,||\mathbb{A}_1(\hat{\mathbf{r}}_{cl}^*)|^2 {}_s\mathcal{F}(\hat{\mathbf{r}}_{cl}^*,\hat{\mathbf{k}})|^2 + ||\mathbb{A}_1(\hat{\mathbf{r}}_{cl}^{*\prime})|^2 {}_s\mathcal{F}(\hat{\mathbf{r}}_{cl}^{*\prime},\hat{\mathbf{k}})|^2] \,. \qquad (73)$$

Equation (73), an elastic version of the optical theorem, is the Rayleigh's results for the identical elastic rigid scatterers corresponding to longitudinal incidences. The complex factors ${}_p\mathcal{F}(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})$ and ${}_s\mathcal{F}(\hat{\mathbf{k}}_0,\hat{\mathbf{r}})$ are defined explicitly in (67) and (69), respectively. The remaining energy theorems and the cases for either the fluid-filled cavity or the perfect elastic sphere can be treated similarly.

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[1] V. Twersky, J. Math. Phys. **16**, 633 (1975).
[2] D. Phanord (unpublished).
[3] D. D. Phanord and N. E. Berger, J. Math. Phys. **31**, 2619 (1990).
[4] R. Solakiewicz and D. D. Phanord, Nuovo Cimento B (to be published).
[5] V. Twersky, J. Acoust. Soc. Am. **53**, 96 (1973).
[6] D. D. Phanord and N. E. Berger, J. Math. Phys. **31**, 2626 (1990).
[7] D. E. Budreck and J. H. Rose, *Review of Progress in Quantitative Nondestructive Evaluation,* Proceedings of the 17th Annual Review (Plenum, New York, 1991).
[8] A. G. Ramm and M. A. Fiddy, Opt. Commun. **56**, 8 (1985).
[9] U. M. Sultangazin and I. Sh. Irekegulov, *Ill-Posed Problems of Mathematical Physics* (Nauka, Moscow, 1984), pp. 143–149.
[10] A. Kazakov, Akademiia Nauk SSSR Doklady **270**, 1100 (1983).
[11] A. G. Ramm, Appl. Math. Lett. **5**, 63 (1992).
[12] J. H. Rose, Ames Laboratory, Ames, IA, Report No. DE83-017683 IS-M-433, CONF-8309135-1 (unpublished).
[13] C. F. Bohren and D. H. Huffman, *Absorption and Scattering of Light by Small Particles* (Wiley, New York, 1983).
[14] P. J. Barratt and W. D. Collins, Proc. Cambridge Philos. Soc. **61**, 969 (1965).
[15] V. D. Kupradze, *Progress in Solid Mechanics. III. Dynamical Problems in Elasticity* (North-Holland, Amsterdam, 1963).
[16] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic, New York, 1949).
[17] V. Twersky, IRE Trans. Antennas Propag. **10**, 737 (1962).
[18] V. Twersky, J. Appl. Phys. **23**, 1099 (1952).
[19] V. Twersky, IRE Trans. Antennas Propag. **4**, 330 (1956).
[20] V. Twersky, J. Math. Phys. **3**, 83 (1962).
[21] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
[22] V. Twersky, J. Appl. Phys. **27**, 1118 (1956).
[23] G. Dassios and K. Kiriaki, Q. Appl. Math. **42**, 225 (1984).
[24] V. Varatharajulu, J. Math. Phys. **18**, 537 (1977).
[25] V. Twersky, J. Math. Phys. **8**, 589 (1967).
[26] D. S. Jones, *Acoustic and Electromagnetic Waves* (Clarendon, Oxford, 1986).
[27] D. D. Phanord, Ph.D. thesis, University of Illinois at Chicago, 1988.
[28] N. G. Einspruch, E. J. Witterholt, and R. Truell, J. Appl. Phys. **31**, 806 (1960).